Nonuniform Flow in a Compressor Due to Asymmetric Tip Clearance

This paper presents an analytical study of flow redistribution in a compressor stage due to asymmetric tip clearance distribution. The entire stage is modeled as an actuator disk and it is assumed that upstream and downstream flow fields are determined by the local tip clearance. The flow is assumed to be inviscid and incompressible. First, an axisymmetric flow model is used to connect upstream and downstream flows. Second, a linear perturbation approximation is used for nonaxisymmetric analysis in which each flow variable is assumed to consist of a mean (axisymmetric value) plus a small perturbation (asymmetric value). Thus, the perturbations in velocity and pressure induced by the tip clearance asymmetry are predicted. Furthermore, rotordynamic effects of such flow non-uniformity are examined as well. [S0889-S04X(00)01404-5]

1 Introduction

Nonaxisymmetric tip clearance degrades both aerodynamic and structural performance of turbomachinery, and the tip clearance asymmetry can have many causes, such as rotor shaft bending, whirling, casing asymmetry, and deformation of components. The effects of rotor tip clearance asymmetry on turbine rotors were initially suggested by Thomas [1] and Alford [2]. They suggested that the variation in efficiency with local clearance would lead to a destabilizing forward whirl-inducing force. This suggestion was experimentally verified by Urlichs [3], Wohlrab [4], and Martinez-Sanchez et al. [5]. Martinez-Sanchez et al. also identified nonaxisymmetric pressure acting radially on the turbine hub as a second source of forcing mechanism in addition to the non-axisymmetric torque initially hypothesized by Thomas and Alford. Analytically, Song and Martinez-Sanchez [6,7] developed an actuator disk model that could accurately predict both nonaxisymmetric torque and pressure effects in turbines.

To examine the effects of compressor tip clearance asymmetry, Horlock and Greitzer [8] and Colding-Jorgensen [9] formulated actuator disk models. Also, Ehrich [10] and Graf et al. [11] developed parallel compressor models. All of these models require compressor performance data as inputs. Therefore, Park [12] developed an analytical model to predict the effects of nonaxisymmetric rotor tip clearance in a single-stage compressor without empiricism. All the models predict a backward whirl-inducing force due to torque asymmetry at the design point. In addition, Park [12] predicts a forward whirl-inducing cross force due to pressure asymmetry.

Until now, attention has been focused only on the effects of rotor tip clearance asymmetry. However, real machines operate with tip clearances in both rotors and stators. Furthermore, the results from a recent experiment conducted in the Low Speed Research Compressor (LSRC) at GE [13] strongly suggest contributions from the stator tip clearance asymmetry.

Therefore, this investigation aims to understand the flow fields and rotordynamic effects in compressors caused by nonaxisymmetry in both rotor and stator tip clearances. The scope of current investigation is limited to the effects of static tip clearance asymmetry in a single stage compressor. An analytical actuator disk approach is used in this investigation.

2 Analytical Model

The modeling approach is similar to the approach of Song and Martinez-Sanchez [6,7]. A two-step process is used to solve for the flow through a compressor stage. First step is an axisymmetric, two-dimensional, meridional plane analysis, or the blade scale analysis. This analysis examines radial redistribution of flow due to axisymmetric rotor and stator tip clearances. The top part of Fig. 1 shows the blade scale view of the stage. Due to tip clearance flows, the radially uniform upstream flow is assumed to split into three streams—“a,” “b,” and “c”—upon going through the inlet guide vane (IGV), rotor, and stator. Streams “a” and “c” are associated with the rotor and stator tip clearances, respectively. Stream “b” is the rest of the passage flow, which has passed through the bladed part of both rotor and stator rows.

The second step is a nonaxisymmetric, two-dimensional, radial plane analysis. The tip clearance asymmetry is a nonuniformity with a length scale on the order of the compressor radius. Therefore, this latter analysis is referred to as the radius scale analysis. It examines the flow redistribution in the azimuthal direction caused by nonaxisymmetric tip clearance distribution. This view is shown schematically in the bottom part of Fig. 1. This analysis is a small perturbation (tip clearance asymmetry) analysis about the mean (axisymmetric) solution provided by the blade scale analysis. Therefore, the results from the blade scale analysis are perturbed to provide connecting conditions across the actuator disk.

The actuator disk in this study consists of an inlet guide vane (IGV) row, a rotor blade row, and a stator blade row. The IGV has full-span blades while rotor and stator have partial span blades. Axial, tangential, and radial directions are denoted by x, y, and z, respectively. On the blade scale, −∞ refers to a location far upstream of the IGV. Near upstream of the IGV is referred to as Station 0. Inlet to the rotor is referred to as Station 1, and the rotor exit is called Station 2. Downstream of the stator row is called Station 3. Far downstream is referred to as +∞. On the radius scale x = 0− and x = 0+ are equivalent to −∞ and +∞ at the blade scale, respectively. The compressor’s rotational speed, absolute velocity, and relative velocity are U, C, and W, respectively. α is the absolute flow angle, and β is the relative flow angle.

The model assumes an inviscid, incompressible flow. The compressor geometry is assumed to be two dimensional at the mean radius values. Also, the flow is assumed to follow the blades perfectly. Thus, effects such as blockage and deviation are not accounted for in this model.

2.1 Tip Scale Analysis. Martinez-Sanchez [14] developed an inviscid tip clearance flow model (Fig. 2) whose predictions agreed with the theory and data of Chen [15]. The tip clearance flow (“jet” in Fig. 2) is modeled as a jet driven by the pressure difference between the pressure and suction sides. This jet then collides with an equal amount of passage (“pass” in Fig. 2) flow...
before rolling up into a vortex. Finally, this tip vortex forms a layer that is underturned relative to the passage flow. For example, at the rotor exit, Stream ‘‘b’’ is the passage flow and Stream ‘‘a’’ is the underturned flow due to the rotor tip clearance.

The turbine tip clearance flow model has been modified for compressors [16], and the compressor tip clearance flow model is shown schematically in Fig. 3. The flow velocities on suction and pressure sides are obtained from the Bernoulli equation

\[ W_{ps} = \sqrt{W_1^2 - \frac{p_{ps} - p_1}{\rho}} \]

(1)

\[ W_{ss} = \sqrt{W_1^2 - \frac{p_{ss} - p_1}{\rho}} \]

(2)

where the subscript 1 refers to the rotor inlet condition. Also,

\[ W_G = \sqrt{2p_{ps} - 2p_{ss}} \rho \]

(3)

Since the flow is assumed to be inviscid, the two streams (‘‘jet’’ and ‘‘ss’’ in Fig. 3) that collide have same total pressure, temperature, and also equal static pressures along their contact line. Therefore, these two streams must have equal velocity magnitudes, and the line OP bisects the angle made by \( \dot{W}_{ps} \) and \( \dot{W}_{ss} \) (Fig. 3). Then

\[
\tan 2\theta = \frac{W_G}{W_{ps}} = \sqrt{\frac{C_{p_{ps}} - C_{p_{ss}}}{1 - C_{p_{ps}}}}
\]

(4)

where \( C_p = (p - p_1)/(\rho W_1^2/2) \). Notice that \( C_{p_{ps}} = C_{p_{ss}} \), and it can be shown that the degree of underturning of the vortex relative to the passage flow can be obtained as

\[
\theta = \cos^{-1} \sqrt{\frac{4}{4 + C_p}}
\]

(5)

where

\[
C_p = ZW \left[ \frac{\cos \beta_1}{\cos \beta_2} \right]^2
\]

(6)

and \( ZW \) refers to the Zweifel coefficient

\[
ZW = 2 \left( \frac{s/c}{H/c} \right) \cos^2 \beta_2 \left( \tan \beta_1 - \tan \beta_2 \right)
\]

(7)

### 2.2 Blade Scale Analysis.

The azimuthal momentum equation for the flow is

\[
\vec{C}_z \cdot \nabla C_y = 0
\]

(8)

where \( \vec{C}_{z} = \vec{C}_{z} + \dot{\vec{K}}_{z} \) is the meridional velocity decoupled from the azimuthal velocity. The azimuthal component of vorticity \( \alpha_y \) = \((\partial C_z/\partial z) - (\partial C_z/\partial x)\) can be described in terms of the stream function \( \Psi \) as

\[
\alpha_y(\Psi) = \nabla^2 \Psi = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}
\]

(9)

The upstream flow is irrotational (\( \alpha_y = 0 \)). Therefore, \( \Psi \) obeys the Laplace equation. Downstream, \( \alpha_y \) is concentrated at the interfaces between Streams ‘‘a’’ ‘‘b’’ and ‘‘c’’ (\( Q_b \) and \( Q_s \) in Fig. 1).

Defining the Bernoulli constant as \( B_1 = P_1/\rho + (1/2)C_{z}^2 \),

\[
\alpha_y = \frac{dB_1}{d\Psi}
\]

(10)

From the definition of \( B_{z,j} \), with the continuity constraint and the assumption of spanwise uniform blading,

\[
B_{z,3} - B_{z,1} = \alpha_y \left( \frac{P_3 - P_1}{\rho} \right)
\]

(11)

Upstream of the stage, \( dB_{z,1} / d\Psi = 0 \) due to the flow’s irrotationality. Then, Eq. (10) becomes

\[
\alpha_{y,3} = \frac{d(B_{z,3} - B_{z,1})}{d\Psi} = \frac{d}{d\Psi} \left( \frac{P_3 - P_1}{\rho} \right)
\]

(12)
and \( \omega \), can be determined from the static enthalpy addition \((P_3 - P_1) / \rho \) by the compressor. From Euler’s equation, the static enthalpy rise is given by

\[
\left( \frac{P_3 - P_1}{\rho} \right) = U(C_{y_2} - C_{y_1}) - \frac{1}{2} \left( C_{x_2}^2 - C_{x_1}^2 \right)
\]

At the IGV exit, tangential velocity is given as

\[
C_{y_1} = C_{x_1} \tan \alpha_i
\]

At the rotor exit, the flow has split into two streams. For the bladed stream (Stream “b”), the tangential velocities at the rotor exit (2) and stator exit (3) are

\[
C_{y_2}^b = U - C_{x_2}^b \tan \beta_2
\]

\[
C_{y_3}^b = C_{x_3}^b \tan \alpha_3
\]

Thus, the pressure rise for Stream “b” is

\[
\left( \frac{P_3 - P_1}{\rho} \right)^b = U(C_{y_2}^b - C_{y_1}) - \frac{1}{2} \left[ \left( C_{x_3}^b + C_{x_2}^b \right)^2 - \left( C_{x_1}^b + C_{y_2}^b \right)^2 \right]
\]

For Stream “a,” the tip scale analysis predicts its rotor exit azimuthal velocity to be

\[
C_{x_2}^a = U - C_{x_2}^a \cos \theta_R \sin (\theta_R + \theta_k) / \cos (\beta_m)
\]

where \( \beta_m \) is the mean flow angle through the rotor and \( \theta_R \) is the undertwisting of the Stream “a” relative to the Stream “b.” Also at the stator exit,

\[
C_{x_3}^a = C_{x_3}^a \tan \alpha_3
\]

Similarly, for Stream “c,” the tangential velocities at the rotor exit (2) and stator exit (3) are given as

\[
C_{y_2}^c = C_{x_2}^c \cos \theta_c \sin (\alpha_m + \theta_s) / \cos (\alpha_m)
\]

\[
C_{y_3}^c = C_{x_3}^c \sin \theta_c
\]

Thus, the pressure rises for Streams “a” and “c” are

\[
\left( \frac{P_3 - P_1}{\rho} \right)^a = U(C_{y_2}^a - C_{y_1}) - \frac{1}{2} \left( C_{x_2}^a + C_{x_1}^a \right)^2 + \left( C_{x_1}^a + C_{y_2}^a \right)^2
\]

and

\[
\left( \frac{P_3 - P_1}{\rho} \right)^c = U(C_{y_2}^c - C_{y_1}) - \frac{1}{2} \left( C_{x_2}^c + C_{x_1}^c \right)^2 - \left( C_{x_1}^c + C_{y_2}^c \right)^2
\]

The third terms on the right-hand side of Eqs. (22) and (23) are the kinetic energy dissipated when the leakage through the tip gaps collides with the passage flow before rolling up into vortices.

To focus on the tip clearance effects, the coordinate system is transformed to the streamline coordinate from the \( z \) coordinate.

Then, the equation for \( \Psi \) becomes

\[
\text{Upstream (} x < 0 \text{)} \quad \nabla^2 \Psi = 0
\]

\[
\text{Downstream (} x > 0 \text{)} \quad \nabla^2 \Psi = Q_R \delta(\Psi - \Psi_{\text{rotor tip}}) + Q_s \delta(\Psi - \Psi_{\text{stator tip}})
\]

where \( Q = J \omega \omega \), \( \omega \) is the speed of the \( \omega \), between streams \( i \) and \( j \), and \( \delta \) is Dirac’s delta function.

The boundary conditions are

\[
\Psi(x, 0) = \Psi(x, H) = C_{i0} H
\]

\[
\Psi(x, 0) = \Psi(x, H) = C_{i0} H
\]

\[
\Psi(x = 0 - , z) = C_{i0} z \quad \frac{\partial \Psi}{\partial x}(x = 0 + , z) = 0
\]

\[
\Psi(1, z) = \Psi(3, z) \quad \frac{\partial \Psi}{\partial z}(1, z) = \frac{\partial \Psi}{\partial z}(3, z) = 0
\]

From the definition of \( B \), and Eqs. (17), (22), and (23), the strengths of vorticity, \( Q_R \) and \( Q_s \), in the shear layers (Fig. 1) are

\[
Q_R = U(C_{y_2}^a - C_{y_1}) - U(C_{y_2}^b - C_{y_1})
\]

\[
- \frac{1}{2} \left( C_{x_2}^a + \left( C_{x_2}^a \sin \theta_R / \cos (\beta_m) \right)^2 - C_{x_2}^b \right)
\]

\[
Q_s = U(C_{y_2}^b - C_{y_1}) - U(C_{y_2}^c - C_{y_1})
\]

\[
- \frac{1}{2} \left( C_{x_2}^b - \left( C_{x_2}^b \sin \theta_R / \cos (\alpha_m) \right)^2 - C_{x_2}^c \right)
\]

Subsequently, the velocities at various axial locations can be determined. At the rotor exit, axial velocities for Streams “a” and “b” are

\[
C_{x_2}^a = C_{x_1} \left( 1 + \frac{q_R}{2} \left( 1 - \lambda_R \right) \right)
\]

\[
C_{x_2}^b = C_{x_1} \left( 1 - \frac{q_R}{2} \lambda_R \right)
\]

where \( q_R = Q_R / C_{i1} \) and \( \lambda_R \) is the nondimensional mass fraction of Stream “a.” Tangential velocities are given in Eqs. (15) and (18).

At the stator exit, the axial velocities are

\[
C_{x_3}^a = C_{x_3} \left( 1 + \frac{q_R}{2} \left( 1 - \lambda_R \right) + \frac{q_s}{2} \lambda_s \right)
\]

\[
C_{x_3}^b = C_{x_3} \left( 1 - \frac{q_s}{2} \lambda_R + \frac{q_s}{2} \lambda_s \right)
\]

\[
C_{x_3}^c = C_{x_3} \left( 1 - \frac{q_s}{2} \left( 1 - \lambda_s \right) - \frac{q_R}{2} \lambda_R \right)
\]

and tangential velocities are given in Eqs. (16), (19), and (21).

Far downstream of the stator, the axial velocities are

\[
C_{x_3}^{a} = C_{x_3} \left( 1 + q_R \left( 1 - \lambda_R \right) + q_s \lambda_s \right)
\]

\[
C_{x_3}^{b} = C_{x_3} \left( 1 - q_R \lambda_R + q_s \lambda_s \right)
\]

\[
C_{x_3}^{c} = C_{x_3} \left( 1 - q_s \left( 1 - \lambda_s \right) - q_R \lambda_R \right)
\]

and tangential velocities are equivalent to those at the stator exit given in Eqs. (16), (19), and (21).

Substituting for velocities in Eqs. (27) and (28) yields two quadratic equations for \( Q_R \) and \( Q_s \) as functions of blade geometry and mass fractions of Streams “a” and “c,” \( \lambda_R \) and \( \lambda_s \).

\[
\left( 1 - \lambda_R \right) \tan^2 \alpha_R \left( \frac{q_R}{2} \right)^2 + 2 \tan^2 \alpha_R \left( \frac{1 + \lambda_s \alpha_s}{2} \right) \left( \frac{q_R}{2} \right)^2
\]

\[
+ \left[ 1 + \frac{q_R}{2} \left( 1 - \lambda_R \right) \right] \left( \sin \theta_R / \cos (\beta_m) \right)^2 + \frac{2}{\Phi} \left( 1 + \frac{q_R}{2} \lambda_R \right) T
\]

\[
- \frac{2}{\Phi} \left( 1 - \frac{q_R}{2} \lambda_R \right) \tan \beta_2 = 0
\]

where \( T = \left( \cos (\theta_R) / \sin (\alpha_m + \theta_R) \right) / \left( \cos (\beta_m) \right) \)

and for the stator
\[
\begin{align*}
\lambda_3^2 (\tan^2 \alpha_3 - G_3) - (1 - 2 \lambda_s) G_3 \left( \frac{q_s}{2} \right)^2 & \\
+ 2 \left[ 2 + \left( 1 - \frac{q_R \lambda_R}{2} \right) \left( \lambda_s (\tan^2 \alpha_3 - G_3) + G_3 \right) \left( \frac{q_s}{2} \right) \right] & \\
+ \left( 1 - \frac{q_R \lambda_R}{2} \right)^2 (\tan^2 \alpha_3 - G_3) = 0
\end{align*}
\]

where \( G_3 = \frac{\sin(\theta_3)}{\cos(\alpha_m)} \) and \( \cos(\theta_3 \sin(\alpha_m + \theta_3) \right)^2 \).

Next, the mass fractions, \( \lambda_R \) and \( \lambda_S \), can be determined from the given tip clearances as

\[
\lambda_R = \frac{4(t_R/H)}{1 - \frac{q_R - q_S \lambda_S}{2} + \sqrt{\left( 1 - \frac{q_R - q_S \lambda_S}{2} \right)^2 - 4 \frac{q_R}{2} H}}
\]

and

\[
\lambda_S = \frac{4(t_S/H)}{1 + \frac{q_S + q_R \lambda_R}{2} + \sqrt{\left( 1 + \frac{q_S + q_R \lambda_R}{2} \right)^2 - 4 \frac{q_S}{2} S}}
\]

Thus, from the prescribed tip clearances \( t_R/H \) and \( t_S/H \), Eqs. (37)–(40) can be solved for \( Q_R, Q_S, \lambda_R, \) and \( \lambda_S \).

Far downstream, after the completion of flow readjustment, the thicknesses of Streams “a” and “c” can be determined as

\[
\Delta_R = \lambda_R \left[ 1 - \frac{q_R}{2} \left( 1 - \lambda_R \right) - \frac{q_S}{2} \lambda_S \right] + \frac{q_S}{2} \left( 1 - \lambda_R \right) + \frac{q_R}{2} \lambda_S
\]

\[
\Delta_S = \lambda_S \left[ 1 + \frac{q_S}{2} \left( 1 - \lambda_S \right) + \frac{q_R}{2} \lambda_R \right] - \frac{q_S}{2} \left( 1 - \lambda_S \right) - \frac{q_R}{2} \lambda_R
\]

Also, the pressure rise across the compressor stage can be determined as

\[
P_{\text{in}} - P_{\text{out}} = U(C_{y,1} - C_{y,2}) + \frac{1}{2} C_{b,1} = \frac{-1}{2} C_{x,1} = \frac{1}{2} C_{b,2}^2
\]

**2.3 Radius Scale Analysis.** The rotor offset, \( e \), is assumed to be much smaller than the blade span. Therefore, the tip clearance distribution is given by

\[
t = \tilde{t} + \text{Re}[\hat{C} e^{i(y/R)}]
\]

where \( \tilde{t} \) is the mean rotor tip gap, and \( y \) is the distance from the maximum tip gap in the azimuthal direction (Fig. 4).

**2.3.1 Upstream Flow.** The irrotational upstream flow is given by

\[
\nabla^2 \phi = 0
\]

where \( \phi \) is the velocity potential. Far upstream, \( C_4(\pm \infty) = \hat{C}_{x,-n} \). Near upstream of the stage at \( x = 0^- \), the axial and tangential velocities are

\[
C_4(0-, y, t) = \text{Re}[\hat{C}_{x,-n} + \hat{K}_6 e^{i(y/R)}]
\]

\[
C_4(x, y, t) = \text{Re}[i\hat{K}_6 e^{i(y/R)}]
\]

where \( \hat{K}_6 \) is the complex amplitude of axial velocity perturbation as the flow approaches the disk. Therefore, the upstream pressure is given by

\[
P(x, y, t) = P(-\infty) - \text{Re}[\rho(\hat{C}_{x,-n}) \hat{K}_6 e^{i(y/R)}]
\]

**2.3.2 Downstream Flow.** Downstream of the stage, the flow consists of three regions: Streams “a,” “b,” and “c.” The continuity equation for each stream can be written as

\[
\frac{\partial \Delta_R}{\partial t} + \frac{\partial \Delta_R}{\partial x} + \frac{\partial \Delta_R}{\partial y} = 0
\]

\[
\frac{\partial \Delta_S}{\partial t} + \frac{\partial \Delta_S}{\partial x} + \frac{\partial \Delta_S}{\partial y} = 0
\]

\[
\frac{\partial (H - \Delta_R - \Delta_3)}{\partial t} + \frac{\partial (H - \Delta_R - \Delta_3)}{\partial x} + \frac{\partial (H - \Delta_R - \Delta_3)}{\partial y} = 0
\]

where \( H \) is annulus height and \( \Delta_R \) and \( \Delta_S \) are given by Eqs. (41) and (42). The momentum equations for the three streams can be written as

\[
\frac{\partial \tilde{C} a, b, c}{\partial t} + (\tilde{C} a, b, c \cdot \nabla) \tilde{C} a, b, c + \frac{1}{\rho} \nabla P = 0
\]

where \( \tilde{C} \) is a two-dimensional velocity with axial and tangential components. Now, each flow parameter can be expressed as

\[
C = \tilde{C} + C'
\]

and

\[
C' = \text{Re}[\hat{C} e^{i\varphi + i(y/R)}]
\]

is a small perturbation about the mean. A homogeneous set of equations for eigenvalues is obtained by substituting for each flow variable and linearizing.
where \( A = \alpha \bar{C}_{i} + i(\bar{C}_{i}/R) \), \( B = \alpha \bar{C}_{b} + i(\bar{C}_{b}/R) \), \( C = \alpha \bar{C}_{c} + i(\bar{C}_{c}/R) \), \( D = \alpha (H - \bar{\Delta}_{R} - \bar{\Delta}_{S}) \), \( E = i(H - \bar{\Delta}_{R} - \bar{\Delta}_{S})/R \).

Then, the nontrivial homogeneous solution is
\[
\begin{pmatrix}
\bar{C}_{i} \\
\bar{C}_{b} \\
\bar{C}_{c} \\
\bar{C}_{x} \\
\bar{C}_{y} \\
\bar{\Delta}_{R} \\
\bar{\Delta}_{S} \\
\bar{P}/\bar{p}
\end{pmatrix}
= \sum_{i=1}^{8} \bar{K}_{i} E_{i} \tag{56}
\]
where \( E_{i} \)'s are eigenvectors, and the complex constants \( \bar{K}_{i} \)'s are to be determined from matching.

### 2.3.3 The Upstream–Downstream Coupling

To connect upstream and downstream flows, the results from the blade scale analysis are used. According to the blade scale analysis, the flow variables depend on the local nondimensional tip clearances, \( \bar{t}_{R}/H \& \bar{t}_{S}/H \), and flow coefficient, \( \Phi \). For example,
\[
\frac{C_{u}}{U} = \frac{C_{u}^{\Phi}}{U} \left( \frac{t_{R}}{H} \right) \left( \frac{t_{S}}{H} \right) \Phi \tag{57}
\]
The downstream and upstream perturbation quantities are determined from the blade scale results as shown below. The axisymmetric blade scale results on the right-hand side of Eq. (58) are perturbed to account for the given geometric nonaxisymmetry. The perturbation solutions then become nonaxisymmetric radius scale results on the left-hand side of Eq. (58). The local flow coefficient, \( \Phi \), is also determined from matching upstream and downstream flows.

### 2.4 Calculation of Rotodynamic Coefficients

From the perturbations in flow variables, rotodynamic excitation forces can be predicted. The tangential force exerted on the compressor by the fluid per azimuthal length is defined as
\[
f_{y} = \lambda_{R} q (C_{y1} - \bar{C}_{y1}) + (1 - \lambda_{R}) q (C_{y1} - \bar{C}_{y1}) \tag{59}
\]
where \( q \) is the local mass flux. The mean and the perturbation of \( f_{y} \) are, respectively
\[
\bar{f}_{y} = \bar{\lambda}_{R} \bar{q} (\bar{C}_{y1} - \bar{C}_{y1}) + (1 - \bar{\lambda}_{R}) \bar{q} (\bar{C}_{y1} - \bar{C}_{y1}) \tag{60}
\]
The perturbation in \( f_{y} \) is almost like the torque variation envisioned by Thomas and Alford. However, they assumed that the flow remains axisymmetric upstream and downstream of the compressor, and, thus, ignored the effects of mass flux perturbation, \( q'/\bar{q} \). However, as Eq. (48) shows, rotor and stator tip clearances do indeed induce azimuthal flow redistribution, and this flow redistribution results in a nonaxisymmetric pressure distribution.
The pressure acting on the rotor hub is approximated as the average of pressures at the inlet and the exit of the rotor, \( \langle P \rangle \).

\[
\langle P \rangle = \frac{P_1 + P_2}{2} = P_1 - \frac{P_1 - P_2}{2}
\]

\[
= P_1 - \frac{1}{2} \rho \left[ (C_{1d}^2) \tan^2 \beta_2 - (U - C_{1s})^2 \right] \tag{62}
\]

\[
\langle P \rangle' = \frac{P_1'}{2} - \frac{1}{2} \left[ C_{1d}^2 C_{1s}^2 \tan^2 \beta_2 + (U - C_{1s}) C_{1s}' \right] \tag{63}
\]

Nonuniform tangential force and nonuniform pressure can thus be obtained from Eqs. (61) and (63). Upon projection onto the \( X \), \( Y \) axes, the total excitation force coefficients, or the rotordynamic stiffness coefficients, are

\[
(\alpha_x + i \beta_x)_{\text{total}} = -\frac{\tilde{f}_y + i \tilde{f}_x}{[\tilde{f}_r, (\epsilon/\epsilon_H)]} \tag{64}
\]

The total coefficients \( (\alpha_{\text{total}}) \), are composed of contributions from tangential force asymmetry \( \alpha_{\text{t}} \) and pressure asymmetry \( \alpha_{\text{p}} \). Forces along the rotor offset are called direct forces and are denoted with a subscript \( X \). Forces perpendicular to the rotor offset are called cross forces and are denoted with a subscript \( Y \).

3 Model Predictions

This section presents the model predictions for the selected baseline compressor and other compressors. Initially, the “baseline” compressor chosen for this study is explained, and the predictions for this compressor are given in the following order. First, the radial flow redistribution induced by axisymmetric rotor and stator tip clearances is presented. Second, the azimuthal flow redistribution due to nonaxisymmetric tip clearances is shown. For both, differences and similarities between the predictions of the new model with rotor and stator clearances (RSC model) and those of the model with only the rotor clearance (RC model) [12] are brought out. Third, rotordynamic coefficients at the design point and off-design points are discussed. Finally, the effects of various compressor designs on rotordynamic stiffness coefficients are presented.

The characteristics of the “baseline” compressor are given in Table 1. The design flow coefficient, reaction, and work coefficient have all been set to 0.5 because they are representative of modern compressors. The tip clearance values of 2 percent of the annulus height have been selected because such value is common in research experiments.

3.1 Blade Scale Predictions. Figure 5 shows the radial profiles of axial velocity, absolute tangential velocity, and relative flow angle at the rotor exit after the flow has split into two streams. The hub and endwall are at \( z/H = 0.0 \) and \( z/H = 1.0 \), respectively. Stream “a” is retarded in the axial direction and underturned in the tangential direction relative to Stream “b.” Figure 6 shows the absolute velocity and flow angle profiles at the stator exit. Now, the flow has split into three streams. Relative to Stream “b,” which goes through both rotor and stator blades, Stream “c” shows characteristics similar to those of Stream “a.” Such results agree with the corresponding predictions from the RC model of Park [12] shown in Figs. 7 and 8. Underturning is due to the effects of the tip leakage flow. The axial momentum defect is caused by flow migration away from the tip clearance where the pressure rise across the compressor stage is sensed more. Such effects increase with increasing tip clearance, and this

![Fig. 5 Radial distributions of axial velocity, absolute tangential velocity, and relative flow angle at rotor exit predicted by the new RSC model](image)

![Fig. 6 Radial distributions of axial velocity, absolute tangential velocity, and absolute flow angle at stator exit predicted by the new RSC model](image)

![Fig. 7 Radial distributions of axial velocity, absolute tangential velocity, and relative flow angle at rotor exit predicted by the RC model of Park [12]](image)
trend agrees with the experimental findings of Hunter and Cumpsty [17].

Comparing Figs. 6 and 8, the obvious difference between the new RSC model and the RC model is Stream '''c,'''' which does not exist in Fig. 8. Thus, the new model can incorporate the effects of stator gap on the flow field. Focusing on Stream '''a,'''' the stream's mass fraction, degree of axial momentum defect, and underturning in Figs. 6 and 8 are virtually identical. This is because the downstream stator tip clearance effect occurs over a length scale on the order of the tip clearance. However, the axial blade spacing is on the order of the blade chord, which is at least a couple of orders of magnitude larger than the tip clearance. Thus, Streams '''a''' and '''c''' are practically decoupled from each other.

**3.2 Radius Scale Predictions**

**3.2.1 Azimuthal Flow Redistribution.** First, the upstream azimuthal flow redistribution induced by tip clearance asymmetry is discussed. Nondimensional velocity and pressure perturbations upstream of the compressor predicted by the RC model and the new RSC model are plotted versus azimuthal location \( \theta \) in Figs. 9 and 10, respectively. The minimum gap is at \( \theta = 0 \) deg and maximum gap is at \( \theta = 180 \) deg. Roughly, the mass flux is higher near the minimum gap in both cases. Again, the higher downstream pressure is “felt” more near the maximum gap. Thus, Streams '''a''' and '''c''' are practically decoupled from each other.

The result is a tangential flow migration away from the larger gap toward the smaller gap. Then from the Bernoulli relation, the pressure decreases as flow accelerates. The magnitudes of both perturbations increase significantly when the stator clearance is introduced. As Eq. (58) shows, the clearance asymmetry acts as the forcing term, which induces azimuthal flow redistribution. Therefore, imposing stator tip clearance asymmetry in addition to the rotor tip clearance asymmetry strengthens the forcing effect. Thus, the flow becomes more nonuniform with rotor and stator tip clearances.

Next, the rotordynamic consequences of such flow redistribution are presented. Tangential force perturbation (also referred to as the torque asymmetry or blade loading variation) is plotted versus \( \theta \) in Fig. 11. Since the force on the compressor by the fluid acts in a direction opposite to the direction of rotation, the mean value of tangential force, \( f / \dot{m} \dot{U} \), is negative. Therefore, according to Fig. 11, the compressor rotor blade is loaded less near the maximum gap. The unloading near the maximum gap occurs mainly because the tip leakage flow rate is higher there. Such prediction has been verified by the experimental data from the GE LSRC [18]. Also, introducing stator clearance asymmetry hardly changes the perturbation in blade loading because the rotor tip leakage flow is practically decoupled from the stator tip clearance.

The perturbation in the rotor region static pressure is plotted versus \( \theta \) in Fig. 12. The pressure has its maximum near the maximum gap (\( \theta = 180 \) deg). Although a similar trend is suggested by the GE’s LSRC data, the corresponding experimental data do not exist yet to confirm this effect in compressors. Unlike the blade loading perturbation, the pressure perturbation is more sensitive to
the addition of stator clearance asymmetry (Fig. 10). In turbines, the pressure asymmetry, predicted by a similar actuator disk model, matched well with experimental data [7].

3.2.2 Design Point Rotodynamic Coefficients. The predicted excitation coefficients from the new RSC model are listed in Table 2 and those from the RC model are given in Table 3. The coefficients due to blade loading variation are \( \alpha_{(w/d)} \)'s and those due to pressure variation are \( \alpha_{(p)} \)'s. Both models predict the following: the blade loading variation induces a negative cross force, which promotes a backward whirl, and a negligible direct force. The pressure effect leads to a positive cross force, which promotes a forward whirl, and a positive direct force. However, the models predict different total coefficients. The new model predicts that the \( \alpha_{(total)} \) is positive because \( \alpha_{(p)} \) is bigger than \( \alpha_{(w/d)} \). Thus, a net positive cross force is predicted. However, the RC model predicts a negligible cross force because the blade loading and pressure effects cancel each other out in \( Y \) direction. Both models predict a positive \( \alpha_{(total)} \) between 0.4 and 0.6.

In comparison, the parallel compressor model of Ehrich [10] can predict only \( \alpha_{(w/d)} \). The model uses the difference between compressor characteristics at different axisymmetric tip clearances to predict torque asymmetry, which is assumed to be in phase with the clearance distribution. However, no pressure information is available for the parallel compressor model to predict pressure asymmetry. Nevertheless, like the new RSC model, the parallel compressor model predicts a negative cross force due to torque asymmetry.

<table>
<thead>
<tr>
<th>Direction</th>
<th>( \alpha_{(w/d)} )</th>
<th>( \alpha_{(p)} )</th>
<th>( \alpha_{(total)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X )</td>
<td>0</td>
<td>+0.4</td>
<td>+0.4</td>
</tr>
<tr>
<td>( Y )</td>
<td>-0.7</td>
<td>+2.1</td>
<td>+1.4</td>
</tr>
</tbody>
</table>

Table 2 Excitation force coefficients for the baseline compressor (\( \Phi_D=0.50, \Psi_D=0.50, R_D=0.50 \)) predicted from the new model (RSC)

<table>
<thead>
<tr>
<th>Direction</th>
<th>( \alpha_{(w/d)} )</th>
<th>( \alpha_{(p)} )</th>
<th>( \alpha_{(total)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( X )</td>
<td>0</td>
<td>+0.6</td>
<td>+0.6</td>
</tr>
<tr>
<td>( Y )</td>
<td>-0.7</td>
<td>+0.6</td>
<td>-0.1</td>
</tr>
</tbody>
</table>

Table 3 Excitation force coefficients for the baseline compressor (\( \Phi_D=0.50, \Psi_D=0.50, R_D=0.50 \)) predicted from the rotor clearance-only (RC) model of Park [12]

3.2.3 Off-Design Point Rotodynamic Coefficients. Figure 13 shows a graph of excitation force coefficients versus the operating flow coefficient. In this case, to model an embedded stage, the IGV has been replaced with a stator row. If a compressor operates below its design flow coefficient, excitation force coefficients increase in magnitude because the amplitudes of flow perturbations are magnified at low \( \Phi \). These trends are similar to those predicted and measured in turbines [7]. Figure 14 shows a graph of the cross force excitation coefficient due to blade loading variation, \( \alpha_{(w/d)} \), plotted versus \( \Phi \). \( \alpha_{(w/d)} \) remains negative but decreases slightly in magnitude as \( \Phi \) increases. This trend has been verified experimentally in the LSRC at General Electric [18].

3.3 Parametric Analysis Predictions. This section presents the predicted effects of compressor design parameters on the excitation force coefficients. The selected parameters are the design flow coefficient, \( \Phi_D \), the design work coefficient, \( \Phi_D \), and the design reaction, \( R_D \). They determine compressor blade angles as shown below.

\[
R_D = \left( \frac{1}{2} \right) \left( \tan \alpha_1 + \tan \beta_2 \right) \Phi_D
\]

\[
\Phi_D = 1 - \Phi_D \left( \tan \alpha_1 - \tan \beta_2 \right)
\]

Thus, a change in the value of one of the parameters changes both rotor and stator blade shapes (i.e., angles), and the effects of vari-

![Fig. 12 Perturbation in the average pressure on rotor hub versus azimuthal angle](image1)

![Fig. 13 Predicted total and pressure rotodynamic coefficients versus operating flow coefficient](image2)

![Fig. 14 Predicted cross rotodynamic coefficients due to blade loading perturbation versus operating flow coefficient](image3)
ous compressor designs can be examined. For parametric analysis, one of the three variables is changed while the other two are held constant at the ‘baseline’ values.

Figure 15 shows variation of \( \alpha_{\text{total}} \) and \( \alpha_{(p)} \) as the design flow coefficient is increased. \( \alpha_{(w,d)} \) is the difference between \( \alpha_{\text{total}} \) and \( \alpha_{(p)} \). For the cross force, \( \alpha_{Y(\text{total})} \) decreases with increasing \( \Phi_D \) primarily because \( \alpha_{Y(p)} \) decreases as the magnitude of azimuthal flow nonuniformity is decreased. Also, \( \alpha_{Y(w,d)} \) remains negative and its magnitude increases with \( \Phi_D \). For the direct force, \( \alpha_{Y(p)} \) dominates over \( \alpha_{Y(w,d)} \) and changes sign as the phase of pressure non-uniformity relative to tip clearance distribution shifts. These trends are similar to those predicted for turbines in Song and Martinez-Sanchez [7].

Figure 16 shows variation of excitation force coefficients as the design work coefficient is increased. In the \( Y \) direction, \( \alpha_{Y(\text{total})} \) increases primarily because \( \alpha_{Y(p)} \) increases in magnitude. In the \( X \) direction, \( \alpha_{X(\text{total})} \) does not change much while \( \alpha_{X(w,d)} \) increases. Overall, increasing the work coefficient is equivalent to strengthening the intensity of discontinuity across the actuator disk. Thus, for a given imposed tip clearance asymmetry, the perturbations in the flow field increase. In addition, the phases of the flow perturbations relative to the clearance distribution also shift.

Figure 17 shows the variation of excitation force coefficients versus the design reaction. As \( R_D \) increases, \( \alpha_{Y(p)} \) does not change much, but \( \alpha_{Y(\text{total})} \) is reduced significantly. This change is due to the decrease in the magnitude of azimuthal flow nonaxi-symmetry. \( \alpha_{(w,d)} \) is relatively insensitive to \( R_D \). Thus, at low design reactions, \( \alpha_{Y(\text{total})} \) is increased.

4 Conclusions

The new conclusions of this study can be summarized as follows:

1. A new analytical model has been developed to examine the effects of nonaxisymmetry in rotor and stator tip clearances on the compressor flow field.
2. The new model has reconfirmed the following previously found trends: (a) radial flow migration away from the tip clearance; (b) azimuthal flow migration towards smaller gap area; and (c) direction of rotordynamic forces that arise due to pressure and torque (i.e., blade loading) asymmetry.
3. In addition, for the baseline compressor, the following conclusions can be drawn.
4. Direct force is mostly due to the pressure asymmetry and is positive.
5. Torque asymmetry results in a negative cross force that, without damping, would promote a backward whirl. However, pressure asymmetry results in a positive cross force, which would promote a forward whirl. The net result is a positive cross force.
6. The dominance of pressure asymmetry effects over those of blade loading asymmetry is due to the introduction of stator clearance asymmetry.
7. The flow associated with the rotor tip clearance is hardly affected by the existence of the downstream stator tip clearance.
8. The pressure asymmetry induced by azimuthal flow redistribution increases significantly in magnitude with the addition of stator tip clearance asymmetry.
9. Operating at below the design flow coefficient increases the magnitude of excitation force coefficients.
10. Finally, from the results of parametric variation about the baseline compressor, the following conclusions can be drawn.
11. High design flow coefficient and high design reaction decrease the magnitudes of excitation force coefficients.
12. High design work coefficient increase the excitation force coefficients’ magnitudes.

Acknowledgments

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Nomenclature

\[ B = \text{Bernoulli constant, m}^2/s^2 \]
\[ c = \text{axial blade chord, m} \]
\[ C_l = \text{lift coefficient per unit span} \]
\[ C_p = \text{pressure coefficient} \]
\[ e = \text{magnitude of rotor offset, m} \]
\[ E_i = \text{eigenvector for downstream perturbations} \]
\[ F_X = \text{lateral force in the direction of the offset, N} \]
\[ F_Y = \text{lateral force perpendicular to the direction of the offset, N} \]
\[ H = \text{annulus height, m} \]
\[ H_b = \text{rotor blade span, m} \]
\[ K_j = \text{complex amplitude of flow perturbations} \]
\[ L = \text{axial rotor hub thickness, m} \]
\[ m = \text{mass flux, kg/s} \]
\[ P = \text{pressure, Pa} \]
\[ Q = \text{strength of shear layer, m}^2/s^2 \]
\[ R = \text{mean compressor radius, m} \]
\[ s = \text{blade pitch, m} \]
\[ t = \text{radial tip clearance, m} \]
\[ U = \text{axial compressor rotational speed at the mean radius, m/s} \]
\[ W = \text{relative velocity, m/s} \]
\[ X = \text{direction along the rotor offset} \]
\[ x = \text{axial direction} \]
\[ Y = \text{direction perpendicular to rotor offset} \]
\[ y = \text{tangential direction} \]
\[ z = \text{radial direction} \]
\[ ZW = \text{Zweifel coefficient} \]
\[ \alpha = \text{absolute flow angle, deg; eigenvalue} \]
\[ \alpha_X = \text{direct excitation force coefficient} \]
\[ \alpha_Y = \text{cross excitation force coefficient} \]
\[ \beta = \text{relative flow angle, deg} \]
\[ \Delta = \text{thickness of underturned layer downstream of actuator disk, m} \]
\[ \phi = \text{upstream velocity potential} \]
\[ \Phi = \text{flow coefficient} \]
\[ \lambda = \text{nondimensional mass fraction of underturned flow} \]
\[ \theta = \text{azimuthal mean angle measured in the direction of rotation from the minimum gap location, deg; angle of underturning relative to passage flow, deg} \]
\[ \rho = \text{density, kg/m}^3 \]
\[ \omega = \text{angular velocity of rotor shaft rotation, s}^{-1} \]
\[ \psi = \text{meridional stream function; work coefficient} \]

Subscripts

\[ 0+ = \text{near downstream of the actuator disk on the radius scale} \]
\[ -\infty = \text{far upstream on the blade scale} \]
\[ 0 = \text{IGV inlet on the blade scale} \]
\[ 1 = \text{rotor inlet on the blade scale} \]
\[ 2 = \text{stator inlet on the blade scale} \]
\[ 3 = \text{stator outlet on the blade scale} \]
\[ +\infty = \text{far downstream on the blade scale} \]
\[ \perp = \text{meridional component} \]

Superscripts

\[ a = \text{the part of downstream flow associated with the rotor tip gap} \]
\[ b = \text{the part of downstream flow that has crossed the bladed part of compressor} \]
\[ c = \text{the part of downstream flow associated with the stator tip gap} \]
\[ ^\prime = \text{nonaxisymmetric perturbation} \]
\[ ^* = \text{azimuthal mean, or axisymmetric value} \]
\[ ^\wedge = \text{complex amplitude} \]

References